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THE MINIMAL GRADED FREE RESOLUTION OF THE UNION OF TWO STAR CONFIGURATIONS IN $\mathbb{P}^n$ AND THE WEAK LEFSCHETZ PROPERTY

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Abstract. We find a graded minimal free resolution of the union of two star configurations $\mathbb{X}$ and $\mathbb{Y}$ (not necessarily linear star configurations) in $\mathbb{P}^n$ of type s and t for $s, t \geq 2$, and $n \geq 3$. As an application, we prove that an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ of two linear star configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^3$ of type s and t has the weak Lefschetz property for $s \geq \lfloor \frac{1}{2}\binom{t}{2} \rfloor$ and $t \geq 2$.

1. Introduction

A star configuration set of points in $\mathbb{P}^2$ (see [5]), which was introduced by Geramita, Migliore, and Sabourin in 2006, will be called a linear star configuration in this paper. Configurations of this type and their natural generalizations to $\mathbb{P}^n$ have been proved to be a very interesting family of points, hypersurfaces, and so on. For example, it is easy to describe their defining ideals algebraically (see [6, 8]). Moreover, the graded Betti numbers and shifts of a graded minimal free resolutions of star configurations in $\mathbb{P}^n$ can be described in terms of the number and the degrees of defining forms in $R = \Bbbk[x_0, x_1, \ldots, x_n]$ (see [8]). In addition, Catalisano, Geramita, Gimigliano, Migliore, Nagel, and Shin [3] have studied star configurations in $\mathbb{P}^n$ to calculate the dimensions of the secant varieties to the varieties of reducible curves (see also [2]). There have been continual efforts, which have further developed the properties of star configurations in $\mathbb{P}^n$ (see [1, 2, 6, 7, 9]).

We briefly recall generic Hilbert function and the weak Lefschetz property. Let $\Bbbk$ be an infinite field of characteristic free and $R =$

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$\mathbb{k}[x_0, x_1, \ldots, x_n]$ be an $(n+1)$-variable polynomial ring over a field $\mathbb{k}$. If I is a homogeneous ideal in R, the numerical function $\mathbf{H}_{R/I}(t) := \dim_{\mathbb{k}} R_t - \dim_{\mathbb{k}} I_t$ is called the *Hilbert function of the ring R/I*. If $I := I_{\mathbb{X}}$ is the ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$, then we denote the Hilbert function of $\mathbb{X}$ by $\mathbf{H}_{\mathbb{X}}(t) := \mathbf{H}_{R/I_{\mathbb{X}}}(t)$. In particular, if $\mathbb{X}$ is a finite set of points in $\mathbb{P}^n$, then we say that $\mathbb{X}$ has *generic Hilbert function* if $\mathbf{H}_{\mathbb{X}}(t) = \min\left\{|\,\mathbb{X}\,|, \binom{t+n}{n}\right\}$ for every $t \geq 0$. In addition, for a finite set $\mathbb{X}$ of points in $\mathbb{P}^n$, we define $\sigma(\mathbb{X}) = \min\left\{i \mid \mathbf{H}_{\mathbb{X}}(i-1) = \mathbf{H}_{\mathbb{X}}(i)\right\}$.

Let R/I be a standard graded Artinian algebra. We say that R/I has *the weak Lefschetz property* if, for a general linear form $L \in R$ and for every $d \geq 0$, the multiplication map by L, $\left[R/I\right]_d \xrightarrow{\times L} \left[R/I\right]_{d+1}$, has maximal rank. In this case, L is said to be a *Lefschetz element.* Over the years, there have been several papers which have devoted to a classification of possible Artinian quotients having the weak Lefschetz property (see [4, 9]). Note that since R/I is Artinian, $A_d = 0$ for $d \gg 0$, and so only a finite number of maps have to be considered. The *strong Lefschetz property* says that for every $i \geq 1$ the multiplication map by L^i, $[R/I]_d \overset{\times L^i}{\to} [R/I]_{d+i}$, has maximal rank for every $d \geq 0$.

The Lefschetz properties for a standard graded Artinian $\mathbb{k}$-algebra are algebraic abstractions introduced by Stanley [12]. The weak Lefschetz property has recently received more attendtion, and is a very fundamental and natural property of Artinian algebras (see [4, 7, 12]).

The goal of this paper is to find a graded minimal free resolution of the union $\mathbb{X} \cup \mathbb{Y}$ of two star configurations $\mathbb{X}$ and $\mathbb{Y}$ (not necessarily linear star configurations) in $\mathbb{P}^n$ of type s, t with $s, t \geq 2$ and $n \geq 3$ (see Theorem 3.1) using the Künneth formula and a mapping cone construction. Furthermore, we show that an Artinian ring $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ of two linear star configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^3$ of type s, t with $s \geq \left\lfloor \frac{1}{2}\binom{t}{2} \right\rfloor$ and $t \geq 2$ has the weak Lefschetz property (see Theorem 4.4).

2. A Graded Minimal Free Resolution of A Star configuration in $\mathbb{P}^n$

We first introduce notions of a star configuration in $\mathbb{P}^n$.

DEFINITION 2.1. Let $R = \mathbb{k}[x_0, x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{k}$. For positive integers r and s with $1 \leq r \leq \min\{n, s\}$, suppose $F_1, \ldots, F_s$ are general forms in R of degrees $d_1, \ldots, d_s$, respectively. We call the variety $\mathbb{X}$ defined by the ideal

$$\bigcap_{1\le i_1<\cdots<i_r\le s}(F_{i_1},\ldots,F_{i_r})$$

a star configuration in $\mathbb{P}^n$ of type (r,s). In particular, if $\mathbb{X}$ is a star configuration in $\mathbb{P}^n$ of type $(2,s)$, then we simply call *a star configuration in* $\mathbb{P}^n$ of type s for short.

Notice that, for $s\ge n$, each n-forms $F_{i_1},\ldots,F_{i_n}$ of s-general forms $F_1,\ldots,F_s$ in R define $d_{i_1}\cdots d_{i_n}$ points in $\mathbb{P}^n$ for each $1\le i_1<\cdots<i_n\le s$. Thus the ideal $\bigcap_{1\le i_1<\cdots<i_n\le s}(F_{i_1},\ldots,F_{i_n})$ defines a finite set $\mathbb{X}$ of points in $\mathbb{P}^n$ with

$$\deg(\mathbb{X})=\sum_{1\le i_1<i_2<\cdots<i_n\le s} d_{i_1}d_{i_2}\cdots d_{i_n}.$$

Furthermore, if $F_1,\ldots,F_s$ are general linear forms in R, then we call $\mathbb{X}$ *a linear star configuration in* $\mathbb{P}^n$ *of type* s, respectively.

Theorem 2.2 ([8, Theorem 3.4]). *Let* $\mathbb{X}$ *be a star configuration in* $\mathbb{P}^n$ *of type* (r,s) *defined by general forms* $F_1,\ldots,F_s$ *in* $R=\Bbbk[x_0,x_1,\ldots,x_n]$ *of degrees* $d_1,d_2,\ldots,d_s$*, where* $2\le r\le\min\{s,n\}$*, and let* $d=d_1+\cdots+d_s$*. Then a graded minimal free resolution of* $I_{\mathbb{X}}$ *is*

$$0\to\mathbb{F}_r^{(r,s)}\to\mathbb{F}_{r-1}^{(r,s)}\to\cdots\to\mathbb{F}_1^{(r,s)}\to I_{\mathbb{X}}\to 0 \tag{2.1}$$

where

$$\begin{aligned}
\mathbb{F}_r^{(r,s)} &= R^{\alpha_r^{(r,s)}}(-d),\\
\mathbb{F}_{r-1}^{(r,s)} &= \textstyle\bigoplus_{1\le i_1\le s} R^{\alpha_{r-1}^{(r,s)}}(-(d-d_{i_1})),\\
&\vdots\\
\mathbb{F}_\ell^{(r,s)} &= \textstyle\bigoplus_{1\le i_1<\cdots<i_{r-\ell}\le s} R^{\alpha_\ell^{(r,s)}}(-(d-(d_{i_1}+\cdots+d_{i_{r-\ell}}))),\\
&\vdots\\
\mathbb{F}_2^{(r,s)} &= \textstyle\bigoplus_{1\le i_1<\cdots<i_{r-2}\le s} R^{\alpha_2^{(r,s)}}(-(d-(d_{i_1}+\cdots+d_{i_{r-2}}))),\\
\mathbb{F}_1^{(r,s)} &= \textstyle\bigoplus_{1\le i_1<\cdots<i_{r-1}\le s} R^{\alpha_1^{(r,s)}}(-(d-(d_{i_1}+\cdots+d_{i_{r-1}}))),
\end{aligned}$$

with

$$\alpha_\ell^{(r,s)}=\binom{s-r+\ell-1}{\ell-1}\quad\textit{and}$$

$$\operatorname{rank}\mathbb{F}_\ell^{(r,s)}=\binom{s-r+\ell-1}{\ell-1}\cdot\binom{s}{r-\ell}$$

for $1\le\ell\le r$*. In particular, the last free module* $\mathbb{F}_r^{(r,s)}$ *has only one shift* d*, i.e., a star configuration* $\mathbb{X}$ *in* $\mathbb{P}^n$ *is level. Furthermore, any star configuration* $\mathbb{X}$ *in* $\mathbb{P}^n$ *is arithmetically Cohen-Macaulay.*

THEOREM 2.3 ([10, Proposition 2.5]). *Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star configurations in $\mathbb{P}^2$ of type s and t defined by general linear forms $L_1, \ldots, L_s$ and $M_1, \ldots, M_t$ in $R = \Bbbk[x_0, x_1, x_2]$ with $s \geq \left\lfloor \frac{1}{2}\binom{t}{2} \right\rfloor$ and $t \geq 2$. Then the union $\mathbb{X} \cup \mathbb{Y}$ of two linear star configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^2$ has generic Hilbert function.*

3. A Graded Minimal Free Resolution of The Union of Two Star Configurations in $\mathbb{P}^n$

In this section, we shall fins a graded minimal free resolution of $R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}})$ where $\mathbb{X}$ and $\mathbb{Y}$ are star configurations in $\mathbb{P}^n$ of type s and t with $s, t \geq 2$.

THEOREM 3.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be star configurations in $\mathbb{P}^n$ of type s and t defined by general forms of degrees $d_1, \ldots, d_s$ and $e_1, \ldots, e_t$ with $s, t \geq 2$. Let $d = d_1 + \cdots + d_s$ and $e = e_1 + \cdots + e_t$. Then, for $n \geq 3$, a graded minimal free resolution of $R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}})$ is*

$$\begin{array}{l} 0 \;\to\; R^{(s-1)(t-1)}(-(d+e)) \;\to\; \begin{bmatrix} \bigoplus_{1 \leq i \leq s} R^{t-1}(-(d+e-d_i)) \\ \oplus \\ \bigoplus_{1 \leq i \leq t} R^{s-1}(-(d+e-e_i)) \end{bmatrix} \\ \quad \to\; \bigoplus_{\substack{1 \leq i \leq s \\ 1 \leq j \leq t}} R(-(d+e-d_i-e_j)) \;\to\; R \;\to\; R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}}) \;\to\; 0. \end{array}$$

In particular, if $\mathbb{X}$ and $\mathbb{Y}$ are linear star configuration in $\mathbb{P}^n$ of type s and t with $s, t \geq 3$ and $n \geq 3$, then the initial degree of $I_{\mathbb{X}} \cap I_{\mathbb{Y}}$ is $s + t - 2$.

Proof. We first recall that

(3.1)

$$\begin{array}{ccccccccc} 0 & \to & R^{s-1}(-d) & \to & \bigoplus_{1 \leq i \leq s} R(-(d-d_i)) & \to & R & \to & R/I_{\mathbb{X}} & \to & 0, \\ 0 & \to & R^{t-1}(-e) & \to & \bigoplus_{1 \leq i \leq t} R(-(e-e_i)) & \to & R & \to & R/I_{\mathbb{Y}} & \to & 0 \end{array}$$

are a graded minimal free resolutions of the Cohen-Macaulay rings $R/I_{\mathbb{X}}$ and $R/I_{\mathbb{Y}}$ of codimension 2, respectively (see Theorem 2.2).

Notice that $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is a Cohen-Macaulay ring of codimension 4 (see [3, Proposition 3.1]), which implies a projective dimension 4. Hence, by a mapping cone construction, the projective dimension of $R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}})$ is 3. So we obtain the following diagram.

(3.2)

$$
\begin{tikzcd}
 & & 0 \arrow[d] \\
0 \arrow[d] & 0 \arrow[d] & \mathbb{F}_3 \arrow[d] \\
\mathbb{F}_3 \arrow[d] & 0 \arrow[d] & \mathbb{F}_2 \arrow[d] \\
\mathbb{F}_2 \arrow[d] & R^{s-1}(-d) \oplus R^{t-1}(-e) \arrow[d] & \left[\mathbb{F}_1 \oplus R^{s-1}(-d) \oplus R^{t-1}(-e)\right] \arrow[d] \\
\mathbb{F}_1 \arrow[d] & \begin{bmatrix} \bigoplus_{1\le i\le s} R(-(d-d_i)) \\ \oplus \\ \bigoplus_{1\le i\le t} R(-(e-e_i)) \end{bmatrix} \arrow[d] & \begin{bmatrix} \bigoplus_{1\le i\le s} R(-(d-d_i)) \\ \oplus \\ \bigoplus_{1\le i\le t} R(-(e-e_i)) \end{bmatrix} \arrow[d] \\
R \arrow[d] & R \oplus R \arrow[d] & R \arrow[d] \\
R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}}) \arrow[r] \arrow[d] & R/I_{\mathbb{X}} \oplus R/I_{\mathbb{Y}} \arrow[r] \arrow[d] & R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \arrow[d] \\
0 & 0 & 0
\end{tikzcd}
$$

where

$$0 \to \mathbb{F}_3 \to \mathbb{F}_2 \to \mathbb{F}_1 \to R \to R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}}) \to 0$$

is a graded minimal free resolution of $R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}})$. Since $n \geq 3$, by Künneth formula (see [3, Theorem 2.14]), a graded minimal free resolution of $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is

(3.3)

$$
\begin{array}{llcl}
0 & \to & R^{(s-1)(t-1)}(-(d+e)) & \to \begin{bmatrix} \bigoplus_{1\le i\le s} R^{t-1}(-(d+e-d_i)) \\ \oplus \\ \bigoplus_{1\le i\le t} R^{s-1}(-(d+e-e_i)) \end{bmatrix} \\
 & \to & \begin{bmatrix} R^{s-1}(-d) \\ \oplus \\ R^{t-1}(-e) \\ \oplus \\ \bigoplus_{\substack{1\le i\le s \\ 1\le j\le t}} R(-(d+e-d_i-e_j)) \end{bmatrix} & \to \begin{bmatrix} \bigoplus_{1\le i\le s} R(-(d-d_i)) \\ \oplus \\ \bigoplus_{1\le i\le t} R(-(e-e_i)) \end{bmatrix} \\
 & \to & R \to R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \to 0. &
\end{array}
$$

By equations 3.2 and 3.3, we have

$$
\begin{aligned}
\mathbb{F}_3 &= R^{(s-1)(t-1)}(-(d+e)), \\
\mathbb{F}_2 &= \bigoplus_{1\le i\le s} R^{t-1}(-(d+e-d_i)) \oplus \bigoplus_{1\le i\le t} R^{s-1}(-(d+e-e_i)), \\
\mathbb{F}_1 &= \bigoplus_{\substack{1\le i\le s \\ 1\le j\le t}} R(-(d+e-d_i-e_j)),
\end{aligned}
$$

as we wished. In particular, if $\mathbb{X}$ and $\mathbb{Y}$ are linear star configurations in $\mathbb{P}^n$ with $n \geq 3$, then it is immediate from a graded minimal free resolution of $R/(I_{\mathbb{X}} \cap I_{\mathbb{Y}})$ that the initial degree of $I_{\mathbb{X}} \cap I_{\mathbb{Y}}$ is $(s+t-2)$. This completes the proof of this theorem. $\square$

4. An Artinian Ring of Codimension 4 and the Weak Lefschetz Property

We shall prove that a graded Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ of two linear star configurations in $\mathbb{P}^3$ of type s and t with $s \geq \lfloor \frac{1}{2}\binom{t}{2} \rfloor$ and $t \geq 2$ has the weak Lefschetz property. There is a useful numerical characterization of Lefschetz elements, which need some notations.

Definition 4.1. Let $\sum_{i\geq 0} a_i t^i$ be a formal power series, where $a_i \in \mathbb{Z}$. Then we define an associated power series with non-negative coefficients by

$$\left|\sum_{i\geq 0} a_i t^i\right|^+ = \sum_{i\geq 0} b_i t^i,$$

where

$$b_i = \begin{cases} a_i, & \text{if } a_j > 0 \text{ for all } j \leq i, \\ 0, & \text{otherwise.} \end{cases}$$

The following lemma is immediate from the definition of the weak Lefschetz property, so we omit the proof here.

Lemma 4.2. *Let A be a standard artinian graded algebra, and let $L \in A$ be a linear form. Then the following conditions are equivalent:*

(a) *L is a Lefschetz element of A.*

(b) *The Hilbert function of A/LA is given by*

$\dim_{\Bbbk}[A/LA]_i = \max\{0,\ \dim_{\Bbbk}[A]_i - \dim_{\Bbbk}[A]_{i-1}\}$ *for all integers i.*

(c) *The Hilbert series $\mathbf{HS}(A/LA)$ of A/LA is*

$$\mathbf{HS}(A/LA) = |(1-t)\cdot \mathbf{HS}(A)|^+ .$$

Lemma 4.3. *Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^2$ of type s and t with $s \geq \lfloor \frac{1}{2}\binom{t}{2} \rfloor$ and $t \geq 2$. Then*

$$\sigma(\mathbb{X}\cup\mathbb{Y}) < (s+t)-1. \tag{4.1}$$

Proof. Recall that the union $\mathbb{X}\cup\mathbb{Y}$ has generic Hilbert function for $s \geq \lfloor \frac{1}{2}\binom{t}{2} \rfloor$ and $t \geq 2$ (see Theorem 2.3). It is enough to show that

$$\deg(\mathbb{X}\cup\mathbb{Y}) = \binom{s}{2} + \binom{t}{2} \leq \binom{(s+t-3)+2}{2}.$$

This holds by a simple calculation, as we wished. $\square$

THEOREM 4.4. *Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star configurations in $\mathbb{P}^3$ of type s and t with $s \geq \lfloor \frac{1}{2}\binom{t}{2} \rfloor$ and $t \geq 2$, and let $R = \Bbbk[x_0, x_1, x_2, x_3]$. Then an Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak Lefschetz property.*

For convenience, we shall use the following notations in the proof of Theorem 4.4.

1. $R = \Bbbk[x_0, x_1, x_2, x_3]$.
2. $S = \Bbbk[x_0, x_1, x_2] \simeq R/(L)$ where L is a general linear form in R.
3. $\mathbb{X}$ and $\mathbb{Y}$ are linear star configurations in $\mathbb{P}^3$ defined by general linear forms in $R = \Bbbk[x_0, x_1, x_2, x_3]$.
4. $\bar{\mathbb{X}}$ and $\bar{\mathbb{Y}}$ are linear star configurations in $\mathbb{P}^2$, where $\bar{\mathbb{X}}$ and $\bar{\mathbb{Y}}$ are obtained from the restriction of $\mathbb{X}$ and $\mathbb{Y}$ by a general hyperplane $\mathbb{H}$, respectively. So we can think of $\bar{\mathbb{X}}$ and $\bar{\mathbb{Y}}$ as two linear star configurations in $\mathbb{H} \cong \mathbb{P}^2$ (see Theorem 2.2).

Proof of Theorem 4.4 . With notations as above, we define $\sigma := \sigma(\bar{\mathbb{X}} \cup \bar{\mathbb{Y}}) < s + t - 1$ (see Lemma 4.3). For every $d \geq \sigma - 1$

(4.2)
$$\mathbf{H}(S/(I_{\bar{\mathbb{X}}} \cap I_{\bar{\mathbb{Y}}}), d) = \mathbf{H}_{\bar{\mathbb{X}}}(d) + \mathbf{H}_{\bar{\mathbb{Y}}}(d) = \deg(\bar{\mathbb{X}}) + \deg(\bar{\mathbb{Y}}) = \binom{s}{2} + \binom{t}{2}.$$

Recall that L is a general linear form in R, and that, by Theorem 2.2, $\mathbb{X}$ and $\mathbb{Y}$ are arithmetically Cohen-Macaulay schemes in $\mathbb{P}^3$ of codimension 2. So for every $d \geq 0$

$$\Delta\mathbf{H}(R/I_{\mathbb{X}}, d) = \mathbf{H}(S/I_{\bar{\mathbb{X}}}, d) \quad \text{and} \quad \Delta\mathbf{H}(R/I_{\mathbb{Y}}, d) = \mathbf{H}(S/I_{\bar{\mathbb{Y}}}, d).$$

Hence we obtain the exact sequence

(4.3)
$$\begin{array}{ccccc} R/[(I_{\mathbb{X}}, L) \cap (I_{\mathbb{Y}}, L)] & \hookrightarrow & R/(I_{\mathbb{X}}, L) \bigoplus R/(I_{\mathbb{Y}}, L) & \twoheadrightarrow & R/(I_{\mathbb{X}} + I_{\mathbb{Y}}, L) \\ \| & & \| & & \| \\ S/(I_{\bar{\mathbb{X}}} \cap I_{\bar{\mathbb{Y}}}) & \hookrightarrow & S/I_{\bar{\mathbb{X}}} \bigoplus S/I_{\bar{\mathbb{Y}}} & \twoheadrightarrow & S/(I_{\bar{\mathbb{X}}} + I_{\bar{\mathbb{Y}}}). \end{array}$$

It is from equations (4.2) and (4.3) that $[R/(I_{\mathbb{X}} + I_{\mathbb{Y}}, L)]_d = 0$ for every $d \geq \sigma - 1$, so the multiplication map by L

$$\left[R/(I_{\mathbb{X}} + I_{\mathbb{Y}})\right]_{d-1} \stackrel{\times L}{\to} \left[R/(I_{\mathbb{X}} + I_{\mathbb{Y}})\right]_d \to [R/(I_{\mathbb{X}} + I_{\mathbb{Y}}, L)]_d = 0$$

is surjective for such d. Hence, by Lemma 4.2, it suffices to show that the multiplication map by L

$$\left[R/(I_{\mathbb{X}} + I_{\mathbb{Y}})\right]_{d-1} \xrightarrow{\times L} \left[R/(I_{\mathbb{X}} + I_{\mathbb{Y}})\right]_d$$

is injective for $d < \sigma - 1$. In other words, $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak Lefschetz property if and only if

(4.4)
$$\Delta(\mathbf{H}(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}),d) = \mathbf{H}(R/(I_{\mathbb{X}}+I_{\mathbb{Y}},L),d) \quad \text{for every} \quad d<\sigma-1.$$

Recall that from equation (4.3)

$$\mathbf{H}(R/(I_{\mathbb{X}}+I_{\mathbb{Y}},L),d) = \mathbf{H}(S/I_{\bar{\mathbb{X}}},d)+\mathbf{H}(S/I_{\bar{\mathbb{Y}}},d)-\mathbf{H}(S/(I_{\bar{\mathbb{X}}}\cap I_{\bar{\mathbb{Y}}}),d)$$

for every $d<\sigma-1$. Furthermore, notice that

- the initial degree of $I_{\mathbb{X}}\cap I_{\mathbb{Y}}$ is $(s+t-2)$ (see Theorem 3.1), and
- $\sigma < s+t-1$ (see Lemma 4.3).

Hence for every $d<\sigma-1<s+t-2$

$$\mathbf{H}(R/(I_{\mathbb{X}}\cap I_{\mathbb{Y}}),d) = \binom{d+3}{3}. \tag{4.5}$$

Using the exact sequence

$$0 \to R/(I_{\mathbb{X}}\cap I_{\mathbb{Y}}) \to R/I_{\mathbb{X}} \bigoplus R/I_{\mathbb{Y}} \to R/(I_{\mathbb{X}}+I_{\mathbb{Y}}) \to 0, \tag{4.6}$$

we obtain that for every $d<\sigma-1$

$$\begin{aligned}
\Delta \quad & \mathbf{H}(R/(I_{\mathbb{X}}+I_{\mathbb{Y}}),d)\\
&= \Delta\mathbf{H}(R/I_{\mathbb{X}},d)+\Delta\mathbf{H}(R/I_{\mathbb{Y}},d)-\Delta\mathbf{H}(R/(I_{\mathbb{X}}\cap I_{\mathbb{Y}}),d)\\
&= \Delta\mathbf{H}(R/I_{\mathbb{X}},d)+\Delta\mathbf{H}(R/I_{\mathbb{Y}},d)-\binom{d+2}{2} \quad \text{(by equation (4.5))}\\
&= \mathbf{H}(S/I_{\bar{\mathbb{X}}},d)+\mathbf{H}(S/I_{\bar{\mathbb{Y}}},d)-\binom{d+2}{2}\\
&= \mathbf{H}(S/I_{\bar{\mathbb{X}}},d)+\mathbf{H}(S/I_{\bar{\mathbb{Y}}},d)-\mathbf{H}(S/I_{\bar{\mathbb{X}}}\cap I_{\bar{\mathbb{Y}}},d)\\
&= \mathbf{H}(S/(I_{\bar{\mathbb{X}}}+I_{\bar{\mathbb{Y}}}),d)\\
&= \mathbf{H}(R/(I_{\mathbb{X}}+I_{\mathbb{Y}},L),d),
\end{aligned}$$

as we wished. This completes the proof of this theorem. □

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